Mathematics Competition

Indiana University of Pennsylvania

2018

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.

2. Indicate the most correct answer to each question on the answer sheet provided by blackening the `bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely ll the space with a heavy black line. If you wish to change the answer, erase your rst mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.

3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.

4. Avoid wild guessing since you are penalized for incorrect answers. If, howev-32(er,)-524(y8rs.)-434

1. The Fibonacci numbers are a sequence of numbers with the pattern that any value in the sequence is found by adding the two previous values. The 1st Fibonacci number is 1. The 2nd Fibonacci number is also 1. The third is found by adding the prior two, so the 3rd Fibonacci number is 2. The 4th Fibonacci number is found by adding $1 + 2$ to get 3. We may continue in this way to get the sequence $1, 1, 2, 3, 5, \ldots$
You may want to write out several terms of this sequence as we will use it again later in the contest on question # 19 and question # 46.

   This is the 55th annual IUP High School Mathematics Competition and 55 is one of the Fibonacci numbers. The true statement is:

   A. 55 is the 9th Fibonacci number
   B. 55 is the 10th Fibonacci number
   C. 55 is the 11th Fibonacci number
   D. 55 is the 12th Fibonacci number
   E. None of these

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2. If we solve for H in the equation $5 = \sqrt{kH}$, the solution is:

   A. $H = \frac{10}{k}$
   B. $H = \frac{\sqrt{5}}{k}$
   C. $H = \frac{5}{2k}$
   D. $H = \frac{k}{25}$
   E. $H = \frac{25}{k}$

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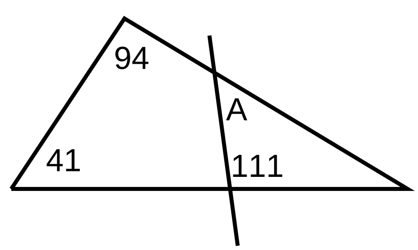
3. In the figure below, the measure of angle A is:



   A. 69
   B. 24
   C. 56
   D. 28
   E. 36

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11. Let $f(x) = ax^2 + bx + c$ be the polynomial having the smallest degree that passes through the points $(1; 8)$, $(\;\;1; 0)$, and $(0; 2)$. The value of abc is equal to:

A. 1

B. 4

C. 6

D. 8

E. None of these

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12. If $\sec(x) = \quad 3$ and $\frac{}{2} < x < \quad$, then the value of $\sin(x)\tan(x)$ is:

A. $\mathrm{p}\,\overline{8}$

B. $\frac{8}{3}$

C. $\quad 1$

D. $\frac{1}{3}$

E. None of these

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13. Suppose $\frac{p}{q + r + s} = \frac{7}{2}$ and $\frac{p}{q + r} = \frac{2}{5}$. Then, the value of $\frac{s}{p}$ is:

A. $\frac{7}{6}$

B. $\frac{11}{14}$

C. $\frac{5}{7}$

D. $\frac{31}{14}$

E. $\frac{14}{11}$

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14. When the repeating decimal number $0{:}36363636 ::: = 0{:}\overline{36}$ is written into simplest reduced fractional form, the sum of the numerator and denominator is:

A. 137

B. 15

C. 9

D. 11

E. None of these

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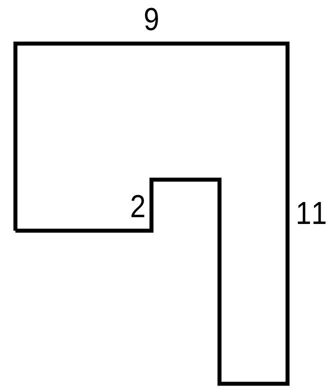
15. The perimeter of the gure given below is:



A. 40 units

B. 42 units

C. 44 units

D. 46 units

E. None of these

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16. The average of m and 9 is x. The average of 2m and 15 is y. The average of 18 and 3m is z. Then, the average of x, y, and z is:

A. m + 6

B. m + 7

C. 2m + 14

D. 3m + 21

E. 6m + 42

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17. If the ratio of 2y x + 6 to 5y + 3x 9 is 2=3, then the ratio of y to x is:

A. $\frac{3}{16}$

B. $\frac{4}{9}$

C. $\frac{3}{8}$

D. $\frac{7}{5}$

E. None of these

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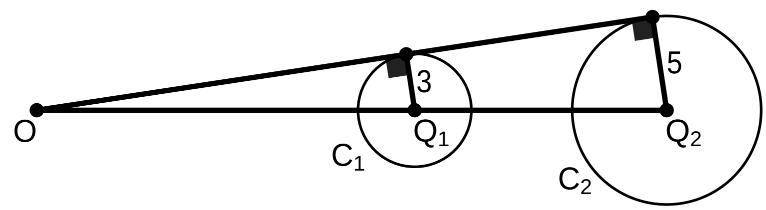
18. In the equation $2\log_b$

22. Given the coordinates $O(0;0)$ and $Q_1(20;0)$, the distance jQ_1Q_2j is:



A. $\frac{40}{3}$

B. 20

C. $\frac{100}{3}$

D. 40

E. None of these

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23. The number of real-valued roots of $\sin^2(x) + \sin(x) \quad 2$ in the interval $[\quad 13;17]$ is:

A. 12

B. 9

C. 5

D. 2

E. 1

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24. Let x be a real number. The unique real number y for which $xy = 1$ is called the multiplicative inverse of x. Now suppose that x is real number that satis es the polynomial equation

$$5x^3 \quad 7x^2 + 4x \quad 3 = 0:$$

The multiplicative inverse of x is equal to:

A. $\quad x$

B. $\dfrac{1}{5x^3 \quad 7x^2 + 4x \quad 3}$

C. $3(5x^2 \quad 7x + 4)$

D. $\dfrac{1}{3}(5x^2 \quad 7x + 4)$

E. None of these

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25. If $f(x) = 2^x$, then the expression $f(x \quad 1) + f(x + 2)$ may be written as:

A. $3f(x)$

B. $f(x + 1)$

C. $\dfrac{5}{2}f(x)$

D. $\dfrac{9}{2}f(x)$

E. $2f(x)$

26. A bacteria colony growing exponentially on a sandwich initially has a population of 10 bacterium. After 2 hours the colony has grown to a population of 6250 bacterium. The amount of time it will take for the bacteria colony to reach a size of 31250 is:

A. $\frac{5}{2}$ hours

B. 5 hours

C. $\frac{7}{2}$ hours

D. 10 hours

E. None of these

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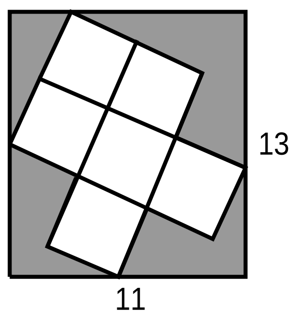
27. Determine the area of the shaded region in the gure below given the following:

The interior 6 rectangles are congruent squares.

Three of the squares placed end-to-end measure 11 units in length.



A. $\frac{187}{3}$ units^2

29. Assuming x and y represent positive real numbers, solving for y in the equation $3x - \sqrt{xy} = \sqrt{x(x+3y)}$ yields:

A. $y = 3x$ or $y = -3x$

B. $y = 0$ or $y = 1$

C. $y = x$ or $y = 16x$

D. $y = x$ or $y = 9x$

E. $y = x^2$

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30. If a cubic polynomial with real coefficients has a root of $3 - i$ and if the product of all of the roots is -5, then the real root of the polynomial is:

A. $3 - i$

B. 5

C. $1/2$

D. -5

E. $-1/2$

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31. New cell phone numbers in a particular city are all of the form 432-555-# # # # where the last four digits may be any number, except no fives may be used. So, there are 6561 of these new phone numbers available. Of these new numbers, the total amount of different phone numbers with exactly four identical digits is:

A. 30

B. 72

C. 96

D. 102

E. None of these

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32. If $y = \cos(x)$ and $0 < x < \frac{\pi}{2}$, then the value of $\sin(2x)$ is:

A. $2y\sqrt{1 - y^2}$

B. $\sqrt{1 + y^2}$

C. $y\sqrt{y^2 + 1}$

D. $(2y^2 + 1)\sqrt{1 - y^2}$

E. $\frac{1 - y^2}{2}$

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33. A building has a height of 30 feet and casts a shadow that is 40 feet long. A person is standing on the top of the building and casts a shadow that is 8 feet long. The height of the person is:

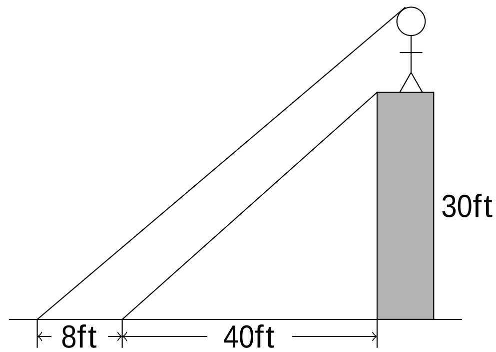
A. 5:5 ft

B. 6 ft

C. 6:25 ft

D. 6:5 ft

E. None of these



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34. The equation $x^3 \quad x^2 + 17x + 87 = 0$ has a solution of $x = \quad 3$. Then the remaining solutions to the equation are:

A. $x = \frac{4 \quad \sqrt{17}}{2}$

B. $x = 2 \quad 5i$

C. $x = \frac{2 \quad 7i}{4}$

D. $x = \frac{3 \quad 2\sqrt{5}}{2}$

E. $x = \quad 4 \quad i\sqrt{7}$

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35. The set B_n represents the set of all binary sequences consisting of n digits. For instance,

$$x = 1001011101001$$

is a binary sequence contained in the set B_{13}. In general, a sequence $x \in B_n$ can be expressed as

$$x = d_1 d_2 \quad d_n;$$

where $d_i = 0$ or $d_i = 1$ for $i = 1; 2; :::; n$. A sequence $x \in B_n$ is called *even* if the sum of its digits is an even number and called *odd* if the sum of its digits is an odd number. The number of odd elements of B_{24} is equal to:

A. 24

B. 2^{12}

C. the number of even elements of B_{24}

D. 2^{24}

E. None of these

36. Suppose the square root of p varies directly as the ratio of q to the square of r. We know $p = 16$ when $q = 24$ and $r = 2$. Then, when $p = 9$ and $q = 2$, the value is r is:

A. $r = 2{=}3$

B. $r = \quad 1$

C. $r = 1$

D. $r = 2$

E. None of these

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37. The value of sin

40. In 1949, Albert Einstein published a popular science article describing his childhood and his rst proof. p(his)]TJ 20.811 -14.4f14w20.811re 16, 2018

43. In a parabolic-type satellite dish, electromagnetic waves enter, hit the satellite dish, and are re ected to a single point P. A wave is said to be re ected if the incoming and outgoing waves both make an angle to the tangent line to the curve. Below, we assume the waves enter parallel to the x-axis and that P

46. Recall the definition of the Fibonacci numbers from question #1. Since this is the 4th month of 2018, suppose we were to take the 2018$^{\text{th}}$ Fibonacci number and divide by 4. We wish to determine the remainder of this division. However, instead of trying to determine such a large Fibonacci number, start at the beginning and write out several numbers in the sequence. Divide each by 4 and look for a pattern in the remainder of each division. From this, we may determine that the remainder when the 2018$^{\text{th}}$ Fibonacci number is divided by 4 is:

A. 0

B. 1

C. 2

D. 3

E. None of these

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47. An equilateral triangle is circumscribed by a circle with radius r. Find r in terms of a, b, c:

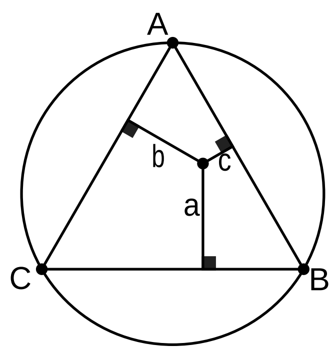
A. $r = \frac{1}{3}(ab + c)$

B. $r = \frac{1}{3}(ab - c)$

C. $r = \frac{2}{3}(abc)$

D. $r = \frac{1}{3}(a + b + c)$

E. None of these



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48. If $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $-1 \le x \le 1$, then π can be expressed as:

A. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

B. $3\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$

C. $4\left(1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \cdots\right)$

D. $3\sqrt{3} - \frac{9}{5} - \frac{27}{7} + \frac{81}{9} - \frac{243}{11} + \cdots$

E. $2\sqrt{3}\left(1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \cdots\right)$

Answer Key

1. B
2. E
3. B
4. B
5. B
6. E
7. E
8. B
18. A
19. C
20. E
21. D
22. A
23. C
24. D
25. D
35. C
36. A
37. D
38. C
39. A
40. B
41. D
42. D